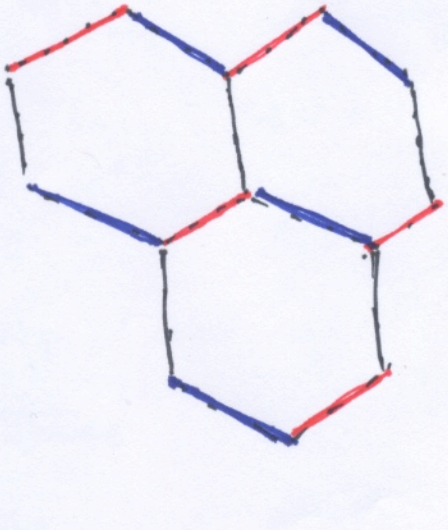
Kitaev's Honeycomb Model

Last time we discussed how a parton construction (e.g. $S = f^\dagger \frac{\sigma}{2} f$ with $f^\dagger f = 1$) can lead to an understanding of emergence of guage theory and anyons at low-energy in 'appropriate' spin-models.

Now, we discuss a model where a parton construction demonstratively leads to a topologically ordered phase (quantum spin-liquid).

The Hilbert space consists of a spin-1/2 d.o.f on each vertex of a honeycomb lattice :



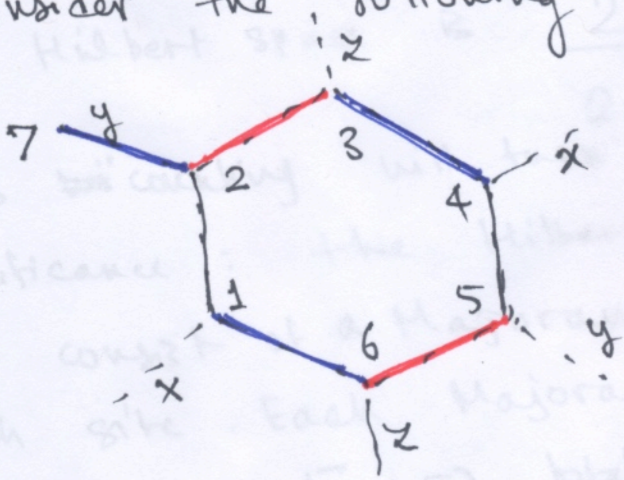
$$H = -J_x \sum_{x\text{-links}} X X - J_y \sum_{y\text{-links}} YY - J_z \sum_{z\text{-links}} ZZ$$

Defining $K_{ij} = X_i X_j$ if $(i,j) \in x$-link

$\qquad\qquad\quad Y_i Y_j$ " y "

$\qquad\qquad\quad Z_i Z_j$ " z " .

Consider the following operator



$$W_p = X_1 Y_2 Z_3 X_4 Y_5 Z_6$$

$$= (Z_1 Z_2)(X_2 X_3)(Y_3 Y_4)(Z_4 Z_5)(X_5 X_6)(Y_1 Y_6)$$

$$= \prod_{\hexagon} K_{ij}$$

As one may readily verify, all operators K_{ij} commute with W_p.

e.g. $K_{12} = Z_1 Z_2$ commute with W_p because W_p involves $X_1 Y_2$. Similarly, ~~for all other~~ K_{27} commute with W_p because the two links that touch K_{27} ~~are~~ $(= Y_2 Y_7)$ involve $Y_2 Z_3$ and $X_1 Y_2$ i.e. they involve only Y_2.

$\Rightarrow$ The Hamiltonian $H = \sum J\, K$ commute with all W_p's. $\Rightarrow$ There are an extensive # of conserved quantities W_ps.

For a fixed value of $\{W_p\}$, the size of the Hilbert space is $\dfrac{2^{\#\text{ sites}}}{2^{\#\text{ plaquettes}}} = 2^{\frac{\#\text{ sites}}{2}}$.

This counting will turn out to have physical significance: the Hilbert space for fixed W_p will consist of a Majorana fermion on each site. Each Majorana fermion has a dimension $\sqrt{2}$ $\Rightarrow$ total dimension $= \sqrt{2}^{\#\text{sites}}$
$= 2^{\#\text{sites}/2}$.

Parton construction

Let's define $X = i\, b^x c$, $Y = i b^y c$, $Z = i b^z c$.
with the constraint $XYZ = -i^3\, b^x b^y b^z c^3 = i$ where b^α and c are Majorana fermions that satisfy $\gamma_i \gamma_j = -\gamma_j \gamma_i$ if $i \neq j$ and $\gamma_i^2 = 1$ where γ can be any of b^α, c.

$$\Rightarrow \boxed{b^x b^y b^z c = 1}$$

This is the analog of $f^\dagger f = 1$ in the alternate parton construction $S = f^\dagger \frac{\sigma}{2} f$.

Let's count physical Hilbert space-dim.

4 Majoranas $\Rightarrow (\sqrt{2})^4 = 2^2$

~~Physics~~ local constraint $b^x b^y b^z c = 1$

$\Rightarrow$ physical Hilbert space $= \frac{2^2}{2} = 2$

$=$ dim. of a spin-$1/2$.

Using this representation,

$$H = -\sum_{(ij)} J_{ij} K_{ij}$$

$$= -\sum_{(ij)} J_{ij} \, i b^\alpha_i c_i \, i b^\alpha_j c_j$$

$$= -\sum_{(ij)} J_{ij} -i(i b^\alpha_i b^\alpha_j) c_i c_j$$

$$= +i \sum_{(ij)} J_{ij} u^\alpha_{ij} c_i c_j$$

where $u_{ij} = i b^\alpha_i b^\alpha_j$.

The most interesting thing about this representation is that u_{ij} commute with H and each other. $\Rightarrow$ One can replace $\hat{u}_{ij} \to u_{ij}$

One can write $H = \frac{i}{4} \sum_{ij} A_{ij} c_i c_j$

where $A_{ij} = 2 J_{ij} u_{ij}$ and each link is counted twice with $u_{ij} = -u_{ji}$

Thus the problem has been reduced to free fermions. The actual ground state is given by

$$|\psi\rangle = \prod_i \left[\frac{1 + b_i^x b_i^y b_j^z c}{2} \right] |\psi_{\text{free ferm ground state}}\rangle$$

Note that U_{ij} anti-commutes with the Operator $D_i = b_i^x b_i^y b_i^z c$, which can be thought of as implementing guage transformation

$$u_{ij} c_i c_j \longrightarrow -u_{ij} c_i c_j$$

However, the guage invariant quantity $\prod_{\hexagon} \hat{U}_{ij}$ commutes with D_i. as well as H. In fact, it equals $W_p = \prod_{\hexagon} K_{ij}$.

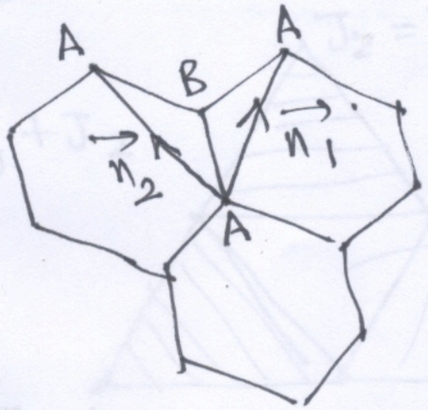
Solution: Although H is parametrized by u_{ij}, the actual (guage-invariant) ground state only depends on W_p.

It turns out that the ground state is given by $W_p = 1$. $\forall\, p$. This follows from a theorem by Lieb. (see Kitaev's paper for the ref.). One can chose a guage $U_{ij} = 1$ when $i \in A$ sublattice, $j \in B$ - sublattice to satisfy $W_p = 1$.

In this guage, H becomes,

$$H = \left[C_A^{+}(q) \quad C_B^{+}(q) \right] \begin{bmatrix} & i f(q) \\ -i f^*(q) & \end{bmatrix} \begin{bmatrix} C_A(-q) \\ C_B(-q) \end{bmatrix}$$

where $C_A(q)$ $C_B(q)$ are Fourier transform of $C_A(r)$, $C_B(r)$. and $f(q) = 2\left[J_x e^{i\vec{q}\cdot\vec{n}_1} + J_y e^{i\vec{q}\cdot\vec{n}_2} + J_z \right]$



Let's set $J_x = J_y = J_z$ for concreteness.

One finds that $f(q)$ is zero at

$q^* = \left(\pm \frac{4\pi}{3\sqrt{3}}, 0 \right)$. One can expand about

q^* and obtain a Dirac-like

dispersion

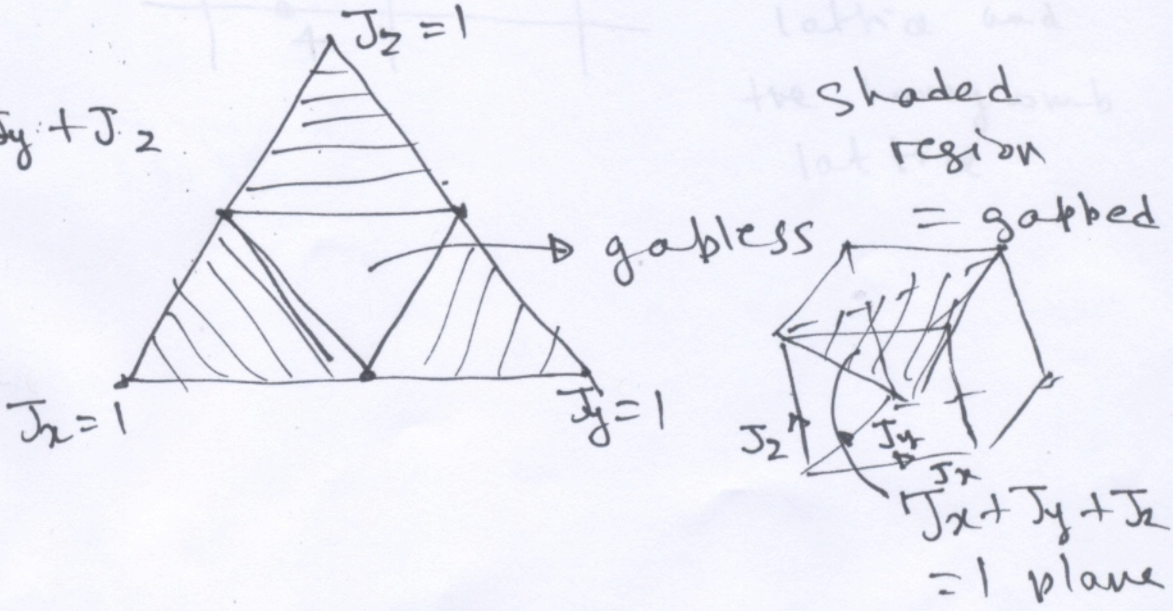
$$\begin{bmatrix} c^\dagger_A(q) & c^\dagger_B(q) \end{bmatrix} \begin{bmatrix} & q_x + i q_y \\ q_x - i q_y & \end{bmatrix} \begin{bmatrix} c_A(q) \\ c_B(q) \end{bmatrix}$$

For several J_x, J_y, J_z, one finds,

with

$J_x + J_y + J_z$

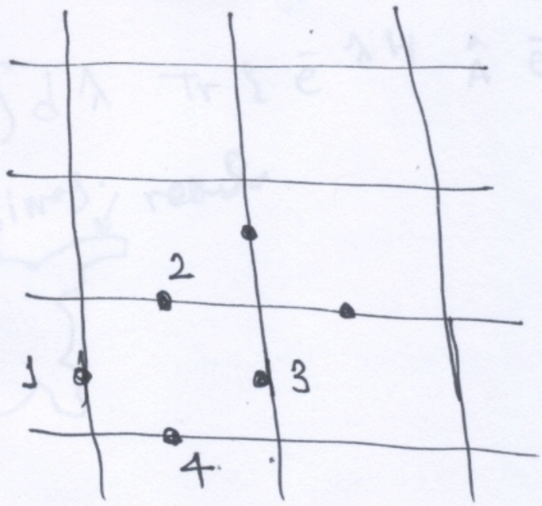
$= 1$



The gapped phase is a gapped $\mathbb{Z}_2$ topologically ordered phase. Infact, one can relate it to toric code.

Consider $J_z \gg J_x, J_y$. As shown in Kitaev's paper, one can derive an effective (low-energy) Hamiltonian:

$$H_{eff} = -\frac{J_x^2 J_y^2}{16 |J_z|^3} \sum_p \sigma_1^y \sigma_2^z \sigma_3^y \sigma_4^z$$



See fig. 6 in Kitaev's paper to see the relation between the rectangular lattice and the honeycomb lattice